Final Exam ISI Bangalore November 2018

Compact Riemann Surfaces 100 Points

Notes.
(a) Justify all your steps. Use only those results that have been proved in class unless you have been asked to prove the same.
(b) $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers.
(c) For a Riemann surface X,
$\mathcal{O}_X$ = the sheaf of holomorphic functions on X,
$\mathcal{E}_X$ = the sheaf of complex-valued C^∞-functions on X,
Ω_X = the sheaf of holomorphic 1-forms on X.
(d) $\mathbb{P}^1$ = the Riemann sphere.

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1. [10 points] For a Riemann surface X, show that the total space $|\mathcal{E}_X|$ is never Hausdorff.

2. [5 points] Let U be an open subset of $\mathbb{C}$ with $z = x + iy$ the coordinate function. Express $\frac{\partial^2}{\partial z\, \partial \bar{z}}$ in terms of linear combinations, products or compositions of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

3. [20 points] Give an example of a Riemann surface X, and a closed form ω of type $(0, 1)$ on X which is not exact. Moreover:
   (i) Find an open cover $\{V_i\}$ of X such that $\omega\big|_{V_i}$ is exact.
   (ii) Find a closed curve $c(t)$ such that $\int_c \omega \neq 0$.

4. [25 points]
   (i) Show that for any Riemann surface X, every point admits a neighbourhood U over which the sheaves $\mathcal{O}_U$ and Ω_U are isomorphic.
   (ii) Prove that $\Omega_{\mathbb{P}^1}(\mathbb{P}^1) = 0$.
   (iii) More generally, prove that if ω is a nonzero meromorphic 1-form on $\mathbb{P}^1$ then the sum of orders of the poles of ω is at least 2.

   [Thus, if you solve (iii), you may immediately deduce (ii) from (iii).]

5. [20 points] Let $\Gamma \subset \mathbb{C}$ be a lattice.
   (i) Let $X \xrightarrow{f} \mathbb{C}/\Gamma$ be a nonconstant holomorphic map of Riemann surfaces. Prove that $\Omega_X(X) \neq 0$.
   (ii) Using (i) or otherwise, deduce that any holomorphic map $\mathbb{P}^1 \to \mathbb{C}/\Gamma$ is constant.
   (iii) Give (without proof) an example of a nonconstant holomorphic map $\mathbb{C}/\Gamma \to \mathbb{P}^1$.

6. [20 points] Let X, Y be compact Riemann surfaces, and let $a_1, \dots, a_n \in X$, $b_1, \dots, b_m \in Y$. Set $X' := X \setminus \{a_1, \dots, a_n\}$, $Y' := Y \setminus \{b_1, \dots, b_m\}$. Show that every isomorphism $f \colon X' \to Y'$ extends to an isomorphism $\widetilde{f} \colon X \to Y$.

[Hint: Let V_j be a chart around b_j and $V_j^* := V_j \setminus \{b_j\}$ the corresponding punctured chart. Prove that there exists a punctured chart U_i^* around each a_i such that $f(U_i^*)$ lies inside some V_j^*.]